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Exact SU(N) Monopole Solutions with Spherical Symmetry

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ABSTRACT

Exact monopole solutions are constructed for an SU(N + 1) gauge theory spontaneously broken by a single Higgs field in the adjoint representation. The solutions saturate the Bogomolny lower bound on the energy and are spherically symmetric with respect to the angular momentum operator $\vec{J} = -i\vec{r} \times \vec{\nabla} + \vec{T}$, where $\vec{T}$ generates the maximal SU(2) subalgebra of SU(N + 1). Our solutions are the most general ones with the above symmetry and contain N real parameters which may be thought of as specifying the nature of the symmetry breaking. When this symmetry breaking is such that the scalar field matrix has repeated eigenvalues it is found that only one of the possible point monopoles has a corresponding finite energy solution saturating the Bogomolny bound.

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I. INTRODUCTION

After the initial work by 't Hooft and Polyakov,[1] much has been learned about the general structure of monopole solutions in arbitrary gauge theories. While the application of topological considerations allowed for a classification of possible monopoles, it did not provide a method for proving existence, or actually constructing solutions. The latter is the aim of the present paper. In the following we consider an SU(N + 1) gauge theory with a single Higgs field in the adjoint representation,[2] and seek solutions which are time independent with $A_o = 0$. A great simplification occurs in the Bogomolny-Prasad-Sommerfield limit,[3,4] in which the symmetry breaking remains but the scalar fields become massless. In this limit any solution to the first order equations $\vec{B} = \vec{D}\Phi$ will saturate the lower bound on the energy

$$E \geq \lim_{r\to\infty} \int \mathrm{Tr}\, \Phi\, \vec{B}\cdot d\vec{S} \qquad . \tag{1.1}$$

We remark that only for certain types of symmetry breaking is the surface integral on the right-hand side topologically conserved, but that the above bound is always valid, since it depends only on integration by parts and the identity $\vec{D}\cdot\vec{B} = 0$.

The solutions we shall find are the natural generalization to SU(N + 1) of the SU(2) Prasad-Sommerfield solution, and are spherically symmetric with respect to $\vec{J} = -i\vec{r}\times\vec{\nabla} + \vec{T}$, where $\vec{T}$ generates the maximal embedding of SU(2) in SU(N + 1). Using the gauge freedom to make the vector field orthogonal to $\hat{r}$, let us write

$$\vec{A} = [\,\vec{M}(r,\hat{r}) - \vec{T}\,] \times \hat{r}/r$$

$$\Phi = \Phi\,(r,\hat{r}) \tag{1.2}$$

where $\vec{M}$ and Φ are unknown matrix functions transforming respectively as a vector and scalar under $\vec{J}$. Because of the spherical symmetry it suffices to evaluate the fields along say the positive z-axis. The Bogomolny equations $D_3 \Phi = B_3$ and $D_\pm \Phi = B_\pm$ then become

$$r^2 \frac{d\Phi}{dr} = \tfrac{1}{2} [M_+, M_-] - T_3 \tag{1.3a}$$

$$\frac{dM_\pm}{dr} = \mp [M_\pm , \Phi] \quad . \tag{1.3b}$$

When T is the maximal SU(2) embedding in SU(N + 1) with $T_3 = \mathrm{diag}(\frac{N}{2}, \frac{N}{2} - 1, \ldots -\frac{N}{2} + 1, -\frac{N}{2})$, it has been shown[5] that the ansatz for the scalar and vector fields may be taken as

$$\Phi = \frac{1}{2} \begin{pmatrix} \phi_1 & & & & \\ & \phi_2 - \phi_1 & & & \\ & & \ddots & & \\ & & & \phi_N - \phi_{N-1} & \\ & & & & -\phi_N \end{pmatrix}$$

$$M_+ = \begin{pmatrix} 0 & a_1 & & & \\ & 0 & a_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & a_N \\ & & & & 0 \end{pmatrix} \tag{1.4}$$

where the ϕ_m and a_m are real radial functions, and $M_- = (M_+)^T$. Substituting (1.4) into the Bogomolny equations (1.3) we obtain

$$r^2 \frac{d\phi_m}{dr} = (a_m)^2 - m\bar{m} \tag{1.5a}$$

$$\frac{da_m}{dr} = (-\tfrac{1}{2}\phi_{m-1} + \phi_m - \tfrac{1}{2}\phi_{m+1})a_m \tag{1.5b}$$

where $1 \leq m \leq N$, and we have defined $\bar{m} = N + 1 - m$ and $\phi_0 = \phi_{N+1} = 0$. Following Ref. 6, we may now solve (1.5b) by introducing N new functions $Q_1, Q_2, \ldots Q_N$ with

$$a_m = \frac{r}{Q_m} [m\bar{m}\, Q_{m-1} Q_{m+1}]^{\frac{1}{2}} \tag{1.6}$$

$$\phi_m = -\frac{d \ln Q_m}{dr} + \frac{m\bar{m}}{r} \tag{1.7}$$

where $Q_1 \equiv Q_{N+1} \equiv 1$. The remaining equation (1.5a) now becomes homogeneous in the Q_m

$$Q_m' Q_m' - Q_m Q_m'' = m\bar{m}\, Q_{m+1} Q_{m-1} \qquad , \tag{1.8}$$

for $m = 1,2,\ldots,N$. Note that in order for (1.6) and (1.7) to be well-defined, the Q_m must never vanish except at the origin. In interpreting the solutions it is useful to observe that if the radial magnetic field B is written in the form $B = \frac{1}{2}\,\mathrm{diag}(B_1, B_2 - B_1, \ldots, B_N - B_{N-1}, -B_N)$ then the B_m are given in terms of the Q_m by

$$B_m = -\frac{d^2 \ln Q_m}{dr^2} - \frac{m\bar{m}}{r^2} \qquad . \tag{1.9}$$

Before embarking on a detailed discussion of explicit solutions, let us make some general remarks concerning the system (1.8) and its associated boundary conditions. Since we are dealing with a set of N coupled second order equations, we should expect, at least locally, that the general solution is a 2N-parameter family. However the requirement that the physical fields be finite at the origin imposes the conditions

$$\lim_{r\to 0} \Phi_m = 0$$

$$\lim_{r\to 0} a_m = \sqrt{m\bar{m}} \tag{1.10}$$

so that from (1.6) and (1.7) it follows that as $r \to 0$

$$Q_m = r^{m\bar{m}} + O(r^{m\bar{m}+1}) \quad . \tag{1.11}$$

Assuming the solution has a power series expansion, let us write

$$Q_m = r^{m\bar{m}} \left(1 + \sum_{n=1}^{\infty} q_m^{(n)} r^n \right) \quad . \tag{1.12}$$

We will call such a solution regular at the origin. Substituting (1.12) into the equations (1.8) one may determine the following:

(1) The coefficients $q_m^{(1)}$ vanish, while for $n \geq 2$ the $q_m^{(n)}$ for $1 \leq m \leq N$ are determined in terms of previously found coefficients by a set of N linear equations. These equations are singly degenerate for $n = 2,3,\ldots,N+1$ in such a way that $q_1^{(2)}$, $q_1^{(3)},\ldots q_1^{(N+1)}$ may be specified arbitrarily, the remaining coefficients then being uniquely determined.

(2) The coefficients always satisfy the relation $q_m^{(n)} = (-1)^n q_{\bar{m}}^{(n)}$ so that all solutions regular at the origin have the reflection property

$$Q_m(-r) = (-1)^{m\bar{m}} Q_{\bar{m}}(r) \quad . \tag{1.13}$$

This property is very useful in constructing explicit solutions.

From (1) above we see that the solutions regular at the origin form an N-parameter subset of the general 2N-parameter solution. At infinity there are also N naturally occurring parameters–the eigenvalues of the traceless matrix Φ. It is reasonable to suppose that the actual solution will map these two N-parameter families into each other, and indeed we shall find this to be the case.

The SU(2) case of the equations (1.8) is trivial and yields the well-known Prasad-Sommerfield solution.[4] In the next section we present the SU(3) example in detail, as it contains all the essential ingredients of the solution for general N which we consider in Section III. The SU(N + 1) solution is naturally described by N + 1 parameters $\alpha_1 \ldots \alpha_{N+1}$ with $\Sigma\alpha_i = 0$ defined by

$$\lim_{r\to\infty} \Phi = -\tfrac{1}{2}\,\mathrm{diag}\,(\alpha_1, \alpha_2, \ldots, \alpha_{N+1}) \quad . \tag{1.14}$$

Most of the previously known solutions[6,7] for $N \geq 2$ correspond to symmetry breakings which could arise from a quartic potential, so that some of the parameters α_i coincide. It is shown that all such solutions may be obtained as suitable limits of our general solution, and some explicit SU(3) and SU(4) examples are given. Section IV contains a brief discussion of the results.

II. SU(3) SOLUTIONS

In the SU(3) case of the system (1.8) there are just two coupled equations

$$Q_1'Q_1' - Q_1Q_1'' = 2Q_2$$

$$Q_2'Q_2' - Q_2Q_2'' = 2Q_1 \quad . \tag{2.1}$$

Let us seek solutions which are regular at the origin so that we may impose the condition $Q_1(-r) = Q_2(r)$ as a simplifying assumption. After some thought one realizes that there are solutions of the form

$$Q_1 = 2\left(A_1e^{\alpha_1 r} + A_2e^{\alpha_2 r} + A_3e^{\alpha_3 r}\right)$$

$$Q_2 = 2\left(A_1e^{-\alpha_1 r} + A_2e^{-\alpha_2 r} + A_3e^{-\alpha_3 r}\right) \quad , \tag{2.2}$$

where the factor 2 has been extracted for convenience and the variables A_i, α_i are constrained by

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \tag{2.3}$$

$$A_iA_j(\alpha_i - \alpha_j)^2 = -A_k \quad . \tag{2.4}$$

Using the three equations (2.4) to solve for the A_i in terms of the α_i leaves a two-parameter set of solutions. However we must still apply the regularity conditions $Q_i \sim r^2$ at the origin. Expanding out the exponentials we find

$$A_1 + A_2 + A_3 = 0$$

$$A_1\alpha_1 + A_2\alpha_2 + A_3\alpha_3 = 0$$

$$A_1\alpha_1^2 + A_2\alpha_2^2 + A_3\alpha_3^2 = 1 \qquad . \tag{2.5}$$

These linear equations have the unique solution

$$A_i = \prod_{j\neq i} (\alpha_i - \alpha_j)^{-1} \qquad , \tag{2.6}$$

which automatically satisfies the constraint (2.4), so we conclude that we have discovered a two parameter set of solutions regular at the origin parametrized by $\alpha_1, \alpha_2, \alpha_3$ with $\Sigma\alpha_i = 0$. A proof that we actually have all the solutions of interest will be sketched for general N in the next section.

Consider now the behavior at infinity. When the α_i are distinct, we choose a labelling such that $\alpha_1 > \alpha_2 > \alpha_3$ and find that as $r \to \infty$

$$\ln Q_1 = \alpha_1 r + O(1)$$

$$\ln Q_2 = -\alpha_3 r + O(1) \qquad . \tag{2.7}$$

Using (1.7), (1.9) yields the asymptotic behavior of the scalar and magnetic fields:

$$\Phi \sim -\tfrac{1}{2}\,\mathrm{diag}\,(\alpha_1, \alpha_2, \alpha_3)$$

$$B \sim -T_3/r^2 \tag{2.8}$$

where $T_3 = \mathrm{diag}\,(1,\ 0,\ -1)$. It is easy to show (for example by considering the functions $Q_i e^{-\alpha_3 r}$ and their first derivatives) that for real α_i the Q_i can never vanish except at the origin and so we have found a meaningful solution for every choice of symmetry breaking with distinct eigenvalues for Φ. A familiar example is $\alpha_1 = 2$, $\alpha_2 = 0$, $\alpha_3 = -2$ giving $Q_1 = Q_2 = \sinh^2 r$, which is merely an embedding of the Prasad-Sommerfield solution in SU(3).

The question remains as to the solutions corresponding to the physically interesting case of symmetry breakings Φ with repeated eigenvalues. Our solutions apparently fail in this limit because the coefficients A_i in (2.6) are divergent when any two α_i coincide. To see that in fact the solutions have a finite limit let us write for example

$$\begin{aligned} \alpha_1 &= 1 + \delta \\ \alpha_2 &= 1 - \delta \\ \alpha_3 &= -2 \end{aligned} \quad . \tag{2.9}$$

Inserting these into the solution (2.2) we obtain

$$Q_1 = \frac{2}{9 - \delta^2} \left\{ \frac{e^r (3 \sinh \delta r - \delta \cosh \delta r)}{\delta} + e^{-2r} \right\} \quad , \tag{2.10}$$

with $Q_2(r) = Q_1(-r)$. We see that the limit $\delta \to 0$ is finite with

$$\begin{aligned} Q_1 &= \frac{2}{9} \left\{ (3r - 1) e^r + e^{-2r} \right\} \\ Q_2 &= \frac{2}{9} \left\{ (-3r - 1) e^{-r} + e^{2r} \right\} \end{aligned} \quad . \tag{2.11}$$

This is a rescaled version of the SU(3) solution found by Bais and Weldon.[6] Note that while the asymptotic behavior of Q_2 is still as in (2.7), that of Q_1 is now different

$$\ln Q_1 = r + \ln r + O(1) \quad ,$$

$$\ln Q_2 = 2r + O(1) \quad , \tag{2.12}$$

so that the asymptotic scalar and magnetic fields have become

$$\Phi \sim \operatorname{diag}(-\tfrac{1}{2}, -\tfrac{1}{2}, 1)$$

$$B \sim \operatorname{diag}(-\tfrac{1}{2}, -\tfrac{1}{2}, 1)/r^2 \quad . \tag{2.13}$$

A similar solution arises when α_2 and α_3 become equal.

Let us note finally that before taking the limit $\delta \to 0$ in (2.11) it is possible to let δ be imaginary and still obtain a real solution. In this case it is easy to see that Q_1 is oscillatory with infinitely many zeros and so cannot correspond to a physically sensible solution.

III. SOLUTION FOR GENERAL N

In this section we will first find a 2N-parameter solution to the N coupled equations (1.8) for $m = 1,2,\ldots,N$, with $Q_0 = Q_{N+1} = 1$. Following the SU(3) example of the previous section let us make the ansatz

$$Q_1 = N! \sum_{i=1}^{N+1} A_i e^{\alpha_i r} \quad , \tag{3.1}$$

where the $2N+2$ parameters α_i, A_i are for the moment arbitrary. Once $Q_0 \equiv 1$ and Q_1 are given, the remaining Q_m (including Q_{N+1}) may be determined uniquely by repeated use of the differential equation (1.8). It is possible to show by induction that, for arbitrary Q_1, the Q_m are homogeneous polynomials of degree m in Q_1 and its derivatives up to $Q_1^{(2m-2)}$. When Q_1 is given by (3.1) this implies that all the Q_m are sums of exponentials, each of which is an m-fold product of exponentials which appear in Q_1. One finds, again by induction, that in fact only terms $\Pi \exp \alpha_i r$ with distinct α_i can occur, and that the Q_m take the explicit form[8]

$$Q_m = (-1)^{\frac{1}{2}m(m-1)} \beta_m \sum_{D_m} \left[\prod_{i \in D_m} (A_i e^{\alpha_i r}) \right] \left[\prod_{\substack{i,j \in D_m \\ i<j}} (\alpha_i - \alpha_j)^2 \right] \tag{3.2}$$

where the constants β_m are given by

$$\beta_m = \left(\prod_{n=1}^{N} n! \right) \Bigg/ \left(\prod_{k=1}^{m-1} k! \prod_{\ell=1}^{\bar{m}-1} \ell! \right) \quad , \tag{3.3}$$

and the sum in (3.2) is over the $\binom{N+1}{m}$ distinct ways that the integers $1,2,\ldots,N+1$ may be divided into two groups D_m and $D_{\bar{m}}$ with m elements in D_m and $\bar{m}$ elements in $D_{\bar{m}}$. Of course the above has achieved nothing unless the Q_{N+1} defined by these equations is equal to unity. However from (3.2), (3.3) we see that Q_{N+1} consists of only a single exponential which may be set to unity by requiring

$$\sum_{i=1}^{N+1} \alpha_i = 0 \tag{3.4}$$

$$(-1)^{\frac{1}{2}N(N+1)}\left[\prod_{i=1}^{N+1} A_i\right]\left[\prod_{j>i} (\alpha_i - \alpha_j)^2\right] = 1 \quad . \tag{3.5}$$

These equations impose two constraints on the variables α_i, A_i in (3.1), leaving a 2N-parameter set which we believe is the general solution to (1.8), although to obtain all possible real solutions, complex values of A_i, α_i must be allowed.

We are only really interested in those solutions which are regular at the origin, i.e. those for which $Q_m \sim r^{m\bar{m}}$ as $r \to 0$. In fact it is only necessary to impose this condition for $m = 1$; it then follows for all m by virtue of the differential equations (1.8). In order that $Q_1 \sim r^N$ at the origin we must have

$$\sum_{i=1}^{N+1} A_i \alpha_i^{\,n} = 0 \ , \ n = 0,1,2,\ldots,N-1 \quad ,$$

$$\sum_{i=1}^{N+1} A_i \alpha_i^{\,N} = 1 \quad . \tag{3.6}$$

Regarding the α_i as given these linear equations have the unique solution

$$A_i = \prod_{j \neq i} (\alpha_i - \alpha_j)^{-1} \quad . \tag{3.7}$$

Note that this choice automatically satisfies the constraint (3.5) so we are left with an N-parameter solution depending on $\alpha_1 \ldots \alpha_{N+1}$ with $\Sigma\alpha_i = 0$. Inserting (3.7) into (3.2) the solution becomes

$$Q_m = \beta_m \sum_{D_m} \left[\prod_{i \in D_m} e^{\alpha_i r}\right]\left[\prod_{\substack{i \in D_m \\ j \in D_{\bar{m}}}} (\alpha_i - \alpha_j)^{-1}\right] \quad . \tag{3.8}$$

Since $\beta_m = \beta_{\bar{m}}$ and $\Sigma\alpha_i = 0$, these solutions have the property $Q_m(-r) = (-1)^{m\bar{m}} Q_{\bar{m}}(r)$, and are the generalization to arbitrary N of the SU(3) example of the previous section.

In order to further investigate the behavior at the origin, it is useful to observe that Q_1 satisfies the linear differential equation

$$\left[\prod_{i=1}^{N+1} \left(\frac{d}{dr} - \alpha_i\right)\right] Q_1 = 0 \qquad . \tag{3.9}$$

Differentiating this equation any number of times and recalling (3.6) one finds that the coefficients of a power series expansion of Q_1 may all be expressed as polynomials in the α_i, and that therefore, despite the form of the coefficients (3.7), the solution is in fact finite when any of the α_i coincide. Furthermore it may be shown that for suitable complex choice of the α_i, the expression (3.8) can reproduce any solution regular at the origin [i.e. any set of complex $q_1^{(2)}, q_1^{(3)}, \ldots, q_1^{(N+1)}$ in (1.12)], and that the solution is real if and only if the α_i are the zeros of a real polynomial. Thus we conclude that the most general real solution of (1.8) regular at the origin is given by (3.8) with the α_i either real or in complex conjugate pairs.

As in SU(3) the physically interesting solutions are those for real α_i. Although we do not have an explicit proof, we believe that in this case the Q_i never vanish except at the origin. To see the behavior at infinity let us first consider distinct α_i with $\alpha_1 > \alpha_2 \ldots > \alpha_{N+1}$. Then the asymptotic behavior of the Q_m is given by

$$\ln Q_m \sim \sum_{i=1}^{m} \alpha_i r + O(1) \qquad . \tag{3.10}$$

Using (1.7), (1.9) we then find that the asymptotic form of the scalar and magnetic fields is

$$\Phi \sim -\tfrac{1}{2}\,\mathrm{diag}\,(\alpha_1, \ldots, \alpha_{N+1})$$

$$B \sim -T_3/r^2 \qquad , \tag{3.11}$$

with $T_3 = \mathrm{diag}(\frac{N}{2}, \frac{N}{2} - 1, \ldots, -\frac{N}{2})$. Consider now the limit in which some of the α_i coincide, say $\alpha_s, \alpha_{s+1}, \ldots, \alpha_{s+n}$ become equal to α. To write down the explicit solution in the general case is rather difficult, but we may easily learn enough to obtain the asymptotic magnetic field. Suppose that as we take the limit, the differences between the eigenvalues $\alpha_s, \ldots, \alpha_{s+n}$ are of some common order δ. Then we see from the solution (3.8) that any term whose exponential contains k of the repeated eigenvalues will contain $k\tilde{k}$ denominators of order δ, where

$$\tilde{k} \equiv n + 1 - k \qquad . \tag{3.12}$$

The limit $\delta \to 0$ can only be finite if the denominator $\delta^{k\tilde{k}}$ is cancelled by the appearance of a term $(\delta r)^{k\tilde{k}}$ in the numerator, so that as $r \to \infty$ the term under consideration acquires a leading factor $r^{k\tilde{k}}$ multiplying the exponential. From this we learn that for $k = 1,2\ldots,n$ the large -r behavior of Q_{s+k-1} is altered such that

$$\ln Q_{s+k-1} = \left(\sum_1^{s-1} \alpha_i + k\,\alpha \right) r + k\tilde{k} \ln r + O(1) \qquad . \tag{3.13}$$

The asymptotic form of the radial magnetic field therefore becomes

$$B \sim (I_3 - T_3)/r^2 \tag{3.14}$$

with

$$I_3 = \mathrm{diag}\,(0, 0, \ldots, \tfrac{1}{2}n, \tfrac{1}{2}n-1, \ldots, -\tfrac{1}{2}n, 0, \ldots, 0) \tag{3.15}$$

where the non-zero entries appear in the s-th through (s+n)-th positions–i.e. the same positions as the equal eigenvalues of ϕ in the asymptotic region. When there is more than one set of equal α's the magnetic field is of the form (3.14) with I_3 a sum of terms of the form (3.15).

We conclude this section with two SU(4) examples. If we write

$$\alpha_1 = 1 + \delta$$

$$\alpha_2 = 1 - \delta$$

$$\alpha_3 = -1 + \delta$$

$$\alpha_4 = -1 - \delta \tag{3.16}$$

and take the limit $\delta \to 0$ we obtain

$$Q_1 = Q_3 = 3(r \cosh r - \sinh r)$$

$$Q_2 = 3(\sinh^2 r - r^2) \tag{3.17}$$

which is the solution found previously by Wilkinson.[7] The asymptotic magnetic field is $B = \mathrm{diag}(-1, -1, 1, 1)/r^2$ and corresponds to $I_3 = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ in (3.15).

Secondly let us consider

$$\alpha_1 = 1 + \delta$$

$$\alpha_2 = 1$$

$$\alpha_3 = 1 - \delta$$

$$\alpha_4 = -3 \qquad . \tag{3.18}$$

Taking the limit $\delta \to 0$ we obtain the SU(4) solution of Bais and Weldon[6]

$$Q_1 = \frac{3}{32}\left\{e^{r}(8r^2 - 4r + 1) - e^{-3r}\right\}$$

$$Q_2 = \frac{3}{16}\left\{e^{2r}(2r^2 - r) + e^{-2r}(2r^2 + r)\right\}$$

$$Q_3 = \frac{3}{32}\left\{e^{3r} - e^{-r}(8r^2 + 4r + 1)\right\} \qquad . \tag{3.19}$$

In this case the asymptotic magnetic field is $B = \mathrm{diag}(-1/2, -1/2, -1/2, 3/2)/r^2$ and corresponds to $I_3 = \mathrm{diag}(1, 0, -1, 0)$ in (3.15).

IV. DISCUSSION

We have in (3.1) to (3.5) exhibited what we believe is the most general solution to the system of N coupled equations (1.8). Applying the appropriate boundary condition at the origin, these led to an N-parameter family (3.8) of non-singular monopoles in SU(N + 1). We should emphasize that we have found only those O(3) symmetric solutions for which $\vec{T}$ is the maximal embedding of SU(2) in SU(N + 1). For $N \geq 2$ there are also embeddings $\vec{T}$ which form reducible representations of SU(2) when viewed as N + 1-dimensional matrices. There are always solutions with the corresponding O(3) symmetry which are merely direct sums of solutions for smaller N, but it is an open question whether there exist any genuine SU(N + 1) solutions with such symmetry.[9]

In spite of the simple and elegant structure of the solution (3.8), to find the explicit expressions in the limiting cases where certain α parameters become equal is non-trivial and deserves further study. In a renormalizable theory with at most quartic scalar couplings, one expects the symmetry breaking to divide the parameters α_i into two groups of equal ones.[10] In the case where one has say $\alpha_i = \bar{k}\,\alpha (1 \leq i \leq k)$ and $\alpha_i = -k\alpha\ (k+1 \leq i \leq N+1)$, it is straightforward to show that

$$Q_1 = \frac{N!}{[(N+1)\alpha]^N}\left[e^{\bar{k}\alpha r}\, L_{k-1}^{-N}(-x) + (-1)^N e^{-k\alpha r} L_{\bar{k}-1}^{-N}(x)\right] \tag{4.1}$$

where $x \equiv (N+1)\alpha r$ and

$$L_m^{-N}(x) = \frac{(-1)^m}{(N-m-1)!}\sum_{\ell=0}^{m}\frac{(N-\ell-1)!}{(m-\ell)!}\,\frac{x^\ell}{\ell!} \tag{4.2}$$

are Laguerre polynomials. In Ref. 6, where the case $k = N$ was solved by an algebraic method, it was found that all the Q_m could be expressed very simply in terms of exponentials multiplied by Laguerre polynomials. We have been unable to show whether the same occurs for $k \neq N$.

Although explicit expressions in the limiting cases present some difficulty, it was possible in (3.14) to compute the form of the magnetic field at large distances. In general the asymptotic form of a monopole solution is always a point monopole with Φ constant and $\vec{D}\,\Phi = 0$. A method of finding all such point solutions was given in Ref. 5, where it was shown that when Φ is a constant matrix Φ_o, the radial magnetic field is always of the form $B = (I_3 - T_3)/r^2$, where I_3 is the 3rd component of an SU(2) embedding $\vec{I}$ with the property that $\vec{I}$ commutes with both $I_3 - T_3$ and Φ_o. When Φ_o has distinct eigenvalues, $\vec{I}$ is necessarily zero, but when some eigenvalues are repeated there is always more than one choice for $\vec{I}$. Our result (3.14), (3.15) shows that of these alternatives, only for the maximal one is there a corresponding finite energy solution of the Bogomolny equations. We may give an intuitive reason for this by noting that the choice (3.15) yields the lowest Coulomb energy in the magnetic field.

It has been pointed out[11] that O(3) symmetric instanton solutions in four-dimensional Euclidean space (t, x, y, z) may be obtained from solutions to the "two-dimensional" version of (1.8):

$$\frac{\partial Q_m}{\partial z}\,\frac{\partial Q_m}{\partial z^*} - Q_m\,\frac{\partial^2 Q_m}{\partial z\,\partial z^*} = m\bar{m}\,Q_{m+1}\,Q_{m-1} \quad , \tag{4.3}$$

where the complex variable z is defined by $z = r + it$. The above equation has a conformal symmetry which leads to the result that if $Q_m(r)$ is any solution of the ordinary differential equation (1.8), then $\left|\frac{1}{g}\,\frac{dg}{dz}\right|^{-m\bar{m}} Q_m(\tfrac{1}{2}\ln g^* g)$ is a solution of (4.3) for any analytic function g(z). It is therefore reasonable to expect that the solution (3.8) will lead to a generalization of the instanton solutions of Ref. 11, a question to which we hope to return elsewhere.

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[2] We use a notation in which the scalar field Φ, vector potential $\vec{A}$, and magnetic field B are expressed as traceless $(N + 1)$-dimensional matrices. Without loss of generality the gauge coupling is taken as unity.

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[8] A direct demonstration that these Q_m satisfy the equations (1.8) is rather hard and shows the power of the result that the Q_m are polynomials in Q_1 and its derivatives.

[9] There are certainly point monopole solutions which are not direct sums. See Reference 5.

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